Multiplicity and Sunspots in General Financial Equilibrium with Portfolio Constraints

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The research question we examine in this paper is the **general equilibrium effects of portfolio constraints**, and especially their **effects on the variability of asset prices**.

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- Can the introduction of portfolio constraints increase the number of equilibria?

- Can portfolio constraints expand the equilibrium set even further, by giving rise to sunspot equilibria?

Results

• Absent the portfolio constraint, the model is a familiar workhorse asset pricing model, with a **unique Pareto efficient equilibrium**.

• With the constraint, the efficient equilibrium still obtains, and
- **Multiplicity**: a finite number of additional (inefficient) equilibria in which the portfolio constraint binds appear,
- **Sunspots**: a continuum of sunspot equilibria (with consumption and prices for commodities varying across them) also appears.

$\Rightarrow$ **sunspots matter**: particular realizations of extrinsic uncertainty affect **real** variables.

Results (cont'd.)

Moreover, we extend our analysis to show that **sunspots also affect asset prices**; i.e., stock price movements need not to be associated with news about economic fundamentals.

This interpretation is consistent with the evidence presented, for example, by Cutler, Poterba, & Summers (1993) and French, & Roll (1986) who show that **news about economic fundamentals explain only a small fraction of the variation in asset prices.**

Related literature

Two main streams of literature,

1. asset pricing with market imperfections (Finance):
- non-uniqueness is either not present or not even considered. Within this literature, our paper is the first to highlight the role of portfolio constraints in generating multiplicity,

2. financial equilibrium (Economics):
- Indeterminacy for nominal asset models (with linear portfolio constraints): Balasko, Cass, & Siconolfi (1989),
- Finite Local Uniqueness for real assets (with general portfolio constraints): Cass, Siconolfi, & Villanacci (2001).

A model of GFE

Leading economic setting:

- two periods: $t = 0, 1$

- two states of nature at $t = 1$: $\omega = u, d$ with probabilities $\pi(\omega)$,

- two goods: $g = 1, 2$

- two real assets (trees): $g = 1, 2$

- return of stock: "g" in terms of good "g" $\delta^g(\omega)$ is set to 1

- two households: $h = 1, 2$

- log-linear preferences
- endowments consist wholly of stocks
- Mr.2 faces a portfolio constraint in his choice of stock 2

GFE is described by equations representing household optimization (first-order conditions, budget constraints, and portfolio constraints) and market clearing conditions.

Extended form equations

1. Solutions to the problem of households:

$$Max_{(c_h, s_h(1))} \left\{ \sum_g \alpha_h^g log c_h^g(0) + \beta \sum_{\omega > 0} \pi(\omega) \sum_g \alpha_h^g log c_h^g(\omega) \right\}$$

subject to	with multipliers
$p(0)(c_h(0) - s_h(0)) + q(s_h(1) - s_h(0)) \leq 0$	$\lambda_h(0)$
$p(\omega)(c_h(\omega) - s_h(1)) \leq 0$, all $\omega > 0$	$\lambda_h(\omega)$
for $h = 2$, $q^2 s_2^2(1) \geq \gamma(q s_2(0))$	μ

2. Market clearing conditions: for commodities (at each of the three spots) and for assets (at spot zero).

Reduced form equations (RFE's)

$p(0) = (1/\beta)(\alpha_1 + \alpha_2\eta(0))$	price of $c(0)$
$p(\omega) = \pi(\omega)(\alpha_1 + \alpha_2\eta(\omega))$, $\omega > 0$	price of $c(\omega)$
$q = \sum_{\omega>0} p(\omega)$	asset prices
$q/\eta(0) - \sum_{\omega>0} \{p(\omega)(1/\eta(\omega))\} + (0, \mu) = 0$	no-arbitrage for $h = 2$
$(1 + 1/\beta) - \sum_{\omega>0} p(\omega)s_1(0) = 0$	BC's ($t = 0$, $h = 1$)
$\pi(\omega)\eta(\omega) - p(\omega)s_1(1) = 0$, $\omega > 0$	BC's ($t = 1$, $h = 1$)
$s_1(1) + s_2(1) = 1$	market clearing

where $\eta(\omega) \equiv \beta/\lambda_h(\omega)$

Asset holdings

We work first with the so-called singular equations:
- BC's ($t = 1$, $h = 1$) and market clearing conditions.

Proposition 1 (portfolio choices): Assume that $a_1 \neq a_2$.
If $(\eta(u), \eta(d)) \gg 0$ and $\eta(u) \neq \eta(d)$, then portfolio choices are given by the unique solution

$s_1(1) = \left(\frac{1-a_2}{a_1-a_2}, \frac{-a_2}{a_1-a_2}\right)$ and $s_2(1) = \left(\frac{-(1-a_1)}{a_1-a_2}, \frac{a_1}{a_1-a_2}\right)$. (A)

If $\eta(u) = \eta(d)$, there is a continuum of optimal portfolio choices, one of which is also (A).

An efficient equilibrium requires $\eta(\omega) = \eta^*$, all ω. Therefore, **at any Pareto allocation, there is indeterminacy in the choice of the asset holdings**.

The regular equations (RE's)

Substituting good and asset pricing eq.'s into no-arbitrage for $h = 2$ and BC's ($t = 0$, $h = 1$), we are left with **three** equations (the so-called RE's) and **four** variables. Let,

- $\xi \equiv (\eta(0), \eta(u), \mu)$ (dependent variables),
- $\theta \equiv \eta(d)$ (independent variable), and
- $\Phi(\xi, \theta) = 0$ (RE's).

Proposition 2 (existence and uniqueness of the efficient equilibrium): If $\alpha_2 s_1(0) \neq 0$ and $\eta^* = \alpha_1 s_2(0)/\alpha_2 s_1(0) > 0$, then $\Phi(\xi, \theta)|_{\mu=0} = 0$ has the unique positive solution $\eta(\omega) = \eta^*$, all ω.

$\Rightarrow$ **there is a unique Pareto allocation, with indeterminacy in the choice of assets and a non-binding portfolio restriction**.

Multiplicity

Proposition 3 (regularity of RE's): Rank $D_\xi \Phi(\xi, \theta) = 3$.

Proposition 4 (global behavior of ξ):

$$D_{\eta(d)}\xi = \begin{bmatrix} D_{\eta(d)}\eta(0) \\ D_{\eta(d)}\eta(u) \\ D_{\eta(d)}\mu \end{bmatrix} = \begin{bmatrix} A(\eta(u) - \eta(d)) \\ -B\frac{\pi(d)\eta(u)}{\pi(u)\eta(d)} \\ C(\eta(d) - \eta(u)) \end{bmatrix},$$

where A, B, C are positive functions of $\eta(d)$.

In particular,

$$\mu \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0 \textit{ according as } \eta(d) \left\{ \begin{array}{c} \neq \\ = \end{array} \right\} \eta^*.$$

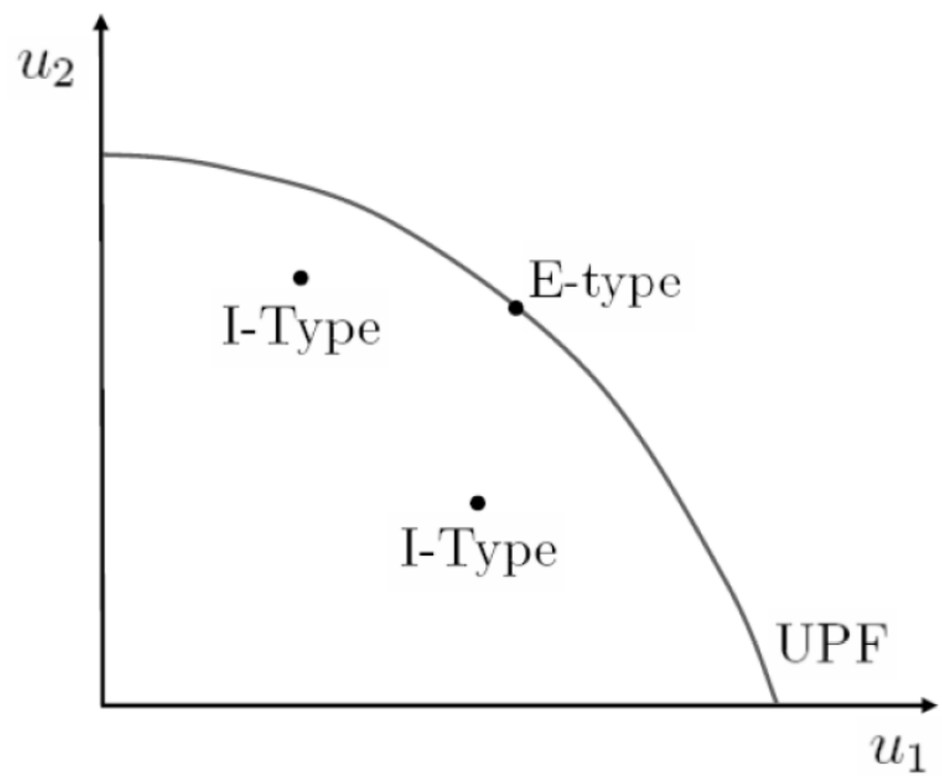
Multiplicity (cont'd)

Result 1 (multiplicity): We have found that:
- the Pareto efficient equilibrium obtains, with the portfolio constraint not binding and a continuum of optimal asset choices,
- two inefficient equilibria exist (bifurcation) with a binding portfolio constraint and a unique optimal choice for asset holdings.

Interpretation of Result 1: There are two types of equilibria, Pareto efficient (E-type) and Pareto inefficient (I-type).
For the E-type, there is a continuum of equilibria, but in which prices and real allocations are identical. Moreover, the asset market is incomplete.
For the I-type, there are exactly two distinct equilibria, but in which the portfolio strategies are identical. Moreover, the asset market is complete.

Multiplicity (cont'd)

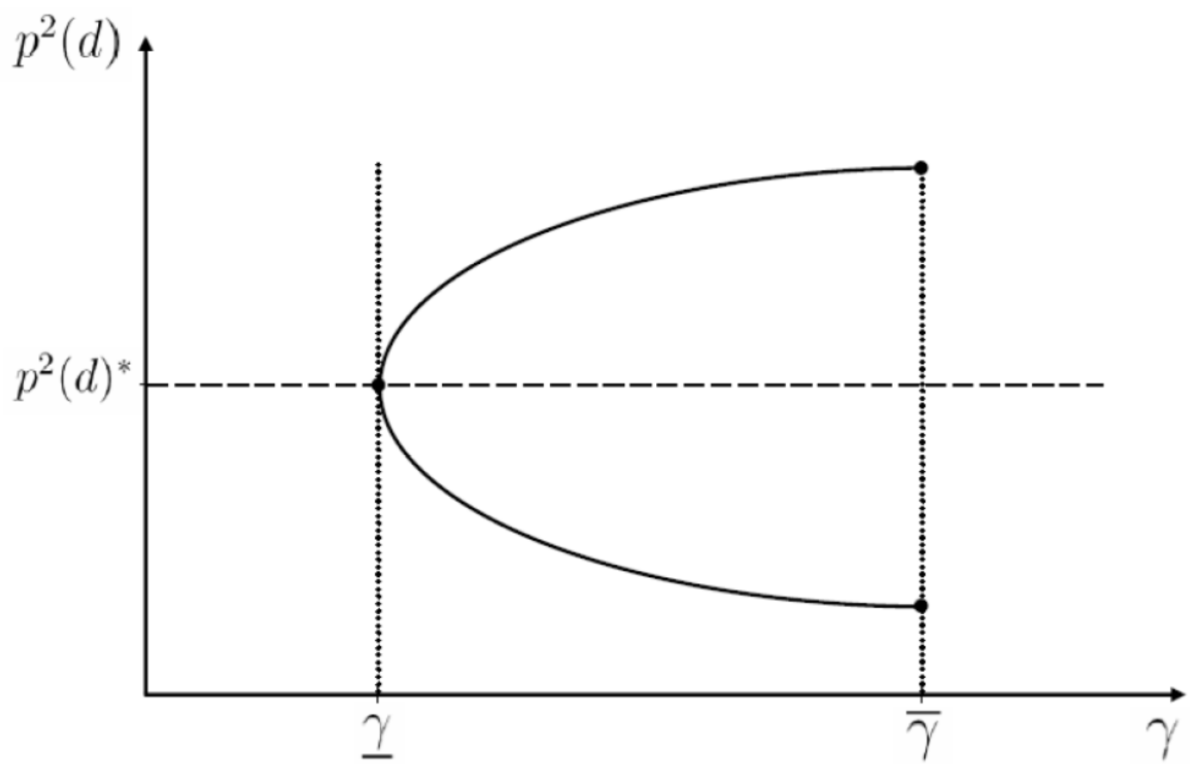
Figure 1: types of equilibria



Multiplicity (cont'd)

Tailoring the portfolio constraint: $q^2 s_2^2(1) \geq \gamma q s_2(0)$

Figure 2: one projection of the equilibrium set



Sunspots

Let's introduce sunspots by additionally incorporating two extrinsic states of nature: $\sigma = G, B$.

Sunspot equilibria occur if for some $\sigma \in \{G, B\}$, $\eta(\sigma, u) \neq \eta(\sigma, d)$.

Proposition 5 (nonsunspot equilibria): Let $\eta(G, u) = \eta(B, u)$ and $\eta(G, d) = \eta(B, d)$. Both, the E-type and I-type of equilibria described in Result 1 still obtain.

Without any restrictions, we can show that there exists a one-dimensional manifold of sunspot equilibria.

Result 2 (indeterminacy of sunspot equilibria): There exists a continuum of sunspot equilibria, with real (but not financial) variables changing across them.

Sunspots: three-period extension

Consider a three-period extension of our model in which:

1. extrinsic uncertainty gets realized first, $\omega_1 = G, B$,
2. households re-trade, and
3. intrinsic uncertainty gets resolved, with $\omega_2 = u, d$.

Assume Mr.2 is constrained in the choice of asset 2 for $t = 0, 1$.

Note that for this economy **the unique efficient equilibrium** (with none of the three portfolio constraints binding) **exists**.

Proposition 6 (asset holdings): If all the portfolio constraints are binding, (A) is the unique optimal choice for asset holdings in both periods. As soon as one constraint does not bind (as it is the case for the efficient equilibrium), indeterminacy arises.

Sunspots: three-period extension (cont'd)

Our "target" sunspot equilibrium has:
(i) $\eta(G,u) = \eta(G,d) = \eta(G)$ (ex-post efficiency after good),
(ii) $\mu(G) = 0$ (portfolio constraint at good node not binding),
(iii)$\eta(B,u) \neq \eta(B,d)$ (ex-post inefficiency after bad),
(ii) $\mu(G) > 0$ (portfolio constraint at bad node binds).

To show existence of our "target" equilibrium, we use the Implicit Function Theorem starting at a different solution.

Proposition 7 (starting point): There exist **regular** sunspot equilibria with the following properties:
$\eta(G,u) = \eta(G,d) = \eta(G)$ and $\mu(G) = 0$, (I)
$\eta(B,u) = \eta(B,d) = \eta(B)$ and $\mu(B) = 0$. (II)
Keeping (I) fixed and perturbing the solutions around the starting point, we get the desired equilibria.

Sunspots: three-period extension (cont'd)

Result 3 (three-period economy): There always exist sunspot equilibria with the following characteristics:

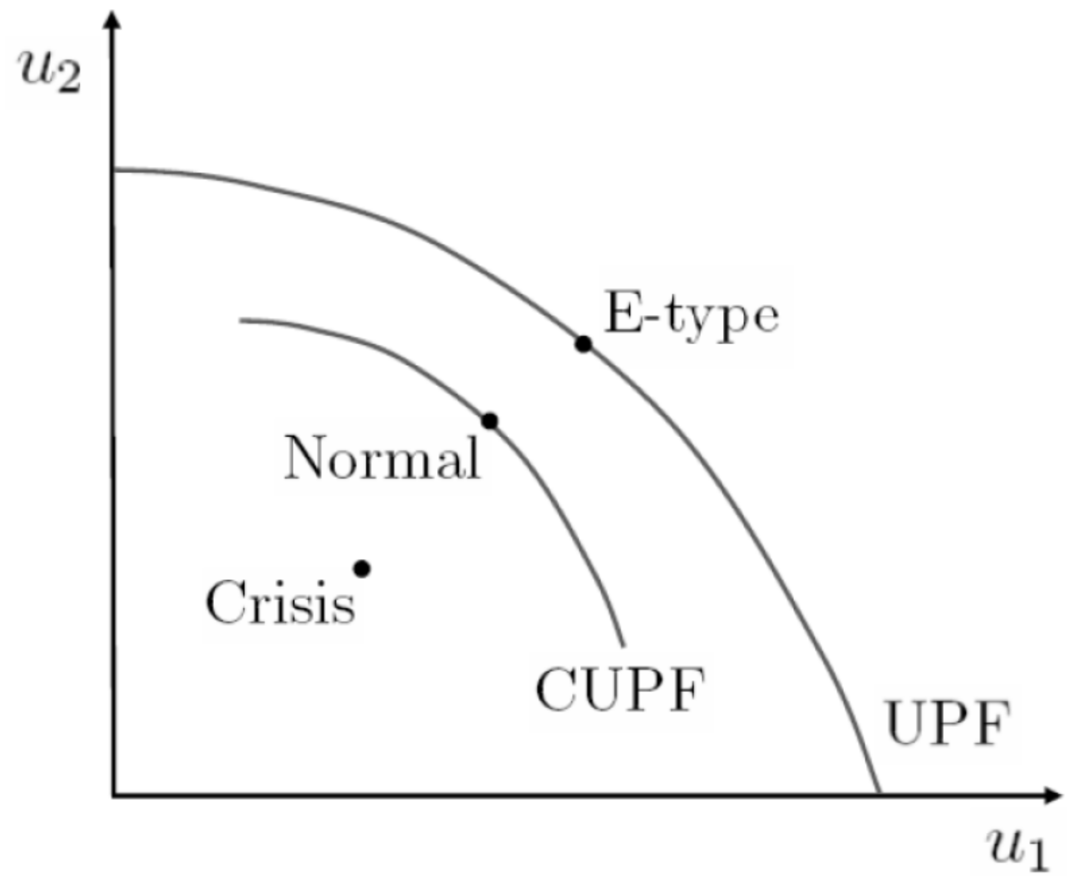
1. for the "good" realization of the sunspot variable,
(i) allocation is ex-post Pareto efficient,
(ii) the associated portfolio constraint does not bind,
(iii) there is indeterminacy in the choice of asset holdings, and
(iv) asset markets –at this spot– are incomplete,

2. for the "bad" realization of the sunspot,
(i) allocation is inefficient,
(ii) the portfolio constraint binds,
(iii) there is a unique optimal choice of asset holdings, and
(iv) asset markets –at this spot– are complete,

3. asset prices and price-dividend ratios vary across "good" and "bad" nodes.

Sunspots: three-period extension (cont'd)

Figure 3: ex-post efficiency



Concluding remarks

We have given an affirmative answer to the question of whether the introduction of portfolio constraints in a GFE model gives rise to new (inefficient) equilibria. We found that portfolio constraints increase the number of equilibria even further, by introducing sunspots and equilibrium indeterminacy.

We have also shown that our model is capable of generating moves in stock prices unrelated to fundamental dividend processes. This may shed light on the apparent inability of empirical studies to link many sharp price movements in stock markets to news about economic fundamentals.

Extensions and further research

Extensions (amenable to local analysis),

1. richer asset structure: bonds and stocks,
2. more than a single portfolio constraint,
3. T periods.

Generalizations,

a. robustness: perturbation of log-linear utilities,
b. robustness: beyond the forest (aka trees),
c. more than two agents,
d. alternative (single) market imperfections.

End of presentation